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Separation of unitary representations of connected Lie groups by their moment sets[☆]

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Abstract

We show that every unitary representation π of a connected Lie group G is characterized up to quasi-equivalence by its complete moment set.

Moreover, irreducible unitary representations π of G are characterized by their moment sets.

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1. Introduction

Let G be a real Lie group with Lie algebra \mathfrak{g} and π a unitary representation of G on a separable Hilbert space \mathcal{H}_π . Note \mathcal{H}_π^∞ the space of \mathcal{C}^∞ vectors for π . Let \mathfrak{g}^* be the dual space of \mathfrak{g} . In [8], Wildberger defined the moment map ψ_π of π as follows.

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For all ξ of $\mathcal{H}_\pi^\infty \setminus \{0\}$, the element $\psi_\pi(\xi)$ in \mathfrak{g}^* is defined by

$$\psi_\pi(\xi)(X) := \frac{1}{i} \frac{\langle d\pi(X)\xi, \xi \rangle_{\mathcal{H}_\pi}}{\langle \xi, \xi \rangle_{\mathcal{H}_\pi}}, \quad X \in \mathfrak{g}. \quad (1)$$

The moment set I_π of the representation π is by definition the closure in \mathfrak{g}^* of the image of the moment map:

$$\psi_\pi : \mathcal{H}_\pi^\infty \setminus \{0\} \longrightarrow \mathfrak{g}^*.$$

Let us suppose π irreducible. Generally, the moment set is the closed convex hull of a co-adjoint orbit in \mathfrak{g}^* (see [3,8]). For instance, if G is an exponential Lie group, I_π is the convex hull of the co-adjoint orbit associated through the Kirillov map to π . Nevertheless, as shown in [8], the moment set does not characterize the representation even for nilpotent connected and simply connected Lie groups.

In [4], Baklouti, Ludwig and Selmi extended the moment map to the dual of the (complex) universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the complexification $\mathfrak{g}_\mathbb{C}$ of \mathfrak{g} as follows:

For all A in $\mathcal{U}(\mathfrak{g})$ and ξ in $\mathcal{H}_\pi^\infty \setminus \{0\}$,

$$\Psi_\pi(\xi)(A) := \Re \left(\frac{1}{i} \frac{\langle d\pi(A)\xi, \xi \rangle_{\mathcal{H}_\pi}}{\langle \xi, \xi \rangle_{\mathcal{H}_\pi}} \right), \quad (2)$$

and define the generalized moment set of π as the convex hull $J(\pi)$ of the image of this generalized moment map Ψ_π .

In [2], it is shown that for all exponential Lie group G , the generalized moment set characterizes the unitary irreducible representations of G . More precisely, we have

Theorem 1.1. *Let $G = \exp \mathfrak{g}$ be an exponential Lie group. Let π and ρ be two unitary irreducible representations of G . Then*

$$\pi \simeq \rho \quad \text{if and only if} \quad J(\pi) = J(\rho).$$

Later on, Abdelmoula et al. [1] extended this result for connected and simply connected type I solvable Lie group of the form $\mathbb{R} \ltimes \mathbb{R}^d$.

In this paper, we introduce the so-called complete moment set $\tilde{J}(\pi)$ (see Definition 2.2) and we show that $\tilde{J}(\pi)$ is the generalized moment set of the representation $\tilde{\pi} = \aleph_0 \pi$. Moreover, if ρ and π are two unitary representations of G having the same generalized moment set ($J(\rho) = J(\pi)$), then their complete moment sets coincide.

Let us say that two unitary representations π and ρ of G are quasi-equivalent if their multiples $\tilde{\pi}$ and $\tilde{\rho}$ are equivalent. Then, we show that the complete moment set $\tilde{J}(\pi)$ characterizes the unitary representation of a connected Lie group G up to quasi-equivalence.

Theorem 1.2. *Let G be a connected Lie group. Let π and ρ be two unitary representations of G in separable Hilbert spaces \mathcal{H}_π , \mathcal{H}_ρ . Then the following are equivalent:*

- (i) π and ρ are quasi-equivalent,
- (ii) $\tilde{J}(\pi) = \tilde{J}(\rho)$.

Finally, in the irreducible case, we get the following characterization:

Theorem 1.3. *Let G be a connected Lie group. Let π and ρ be two irreducible unitary representations of G . Then the following are equivalent:*

- (i) π and ρ are equivalent,
- (ii) $\tilde{J}(\pi) = \tilde{J}(\rho)$,
- (iii) $J(\pi) = J(\rho)$.

In these proofs we use the fact that the spaces \mathcal{H}_π and \mathcal{H}_ρ contain many analytic vectors. The property that $\tilde{J}(\pi) = \tilde{J}(\rho)$ allows us to write a coefficient function γ_ξ^π associated to π for an analytic vector ξ as a converging sum of coefficients $\sum_{i \in \mathbb{N}} \gamma_{\eta_i}^\rho$ with analytic vectors η_i associated to \mathcal{H}_ρ . This then implies the existence of a non-trivial intertwining operator between $\tilde{\pi}$ and $\tilde{\rho}$. The quasi-equivalence of π and ρ follows. Moreover, if π and ρ are irreducible, we can directly define with this method an intertwining operator between \mathcal{H}_π and \mathcal{H}_ρ .

2. Generalized and complete moment sets

In this paper, G is a connected Lie group and every unitary representation π of G is supposed to be carried by a separable Hilbert space \mathcal{H}_π . Let π such a representation of G .

Definition 2.1. The generalized moment map Ψ_π of π associates to each C^∞ vector ξ for π the real linear form on the complex enveloping algebra $\mathcal{U}(\mathfrak{g})$ of \mathfrak{g} defined by

$$\Psi_\pi(\xi)(A) := \Re \left(\frac{1}{i} \frac{\langle d\pi(A)\xi, \xi \rangle_{\mathcal{H}_\pi}}{\langle \xi, \xi \rangle_{\mathcal{H}_\pi}} \right).$$

The generalized moment set $J(\pi)$ of π is then the convex hull of the image of Ψ_π in the real dual $\mathcal{U}(\mathfrak{g})_{\mathbb{R}}^*$ of $\mathcal{U}(\mathfrak{g})$.

In this paper we shall study a new moment set for π , called the complete moment set.

Definition 2.2. Let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of C^∞ vectors for π such that

- (1) $\sum_{n \in \mathbb{N}} \|\xi_n\|_{\mathcal{H}_\pi}^2 = 1$,
- (2) $\sum_{n \in \mathbb{N}} \|d\pi(A)\xi_n\|_{\mathcal{H}_\pi}^2 < \infty$, for all A in $\mathcal{U}(\mathfrak{g})$.

The complete moment mapping $\tilde{\Psi}_\pi$ associates to the sequence (ξ_n) the \mathbb{R} -linear form

$$\tilde{\Psi}_\pi((\xi_n))(A) = \Re \left(\frac{1}{i} \sum_{n \in \mathbb{N}} \langle d\pi(A)\xi_n, \xi_n \rangle_{\mathcal{H}_\pi} \right).$$

The complete moment set $\tilde{J}(\pi)$ of π is then the image of $\tilde{\Psi}_\pi$ in the real dual $\mathcal{U}(\mathfrak{g})_{\mathbb{R}}^*$ of $\mathcal{U}(\mathfrak{g})$.

Hypothesis (2) in our definition guarantees the convergence of the series defining $\tilde{\Psi}_\pi((\xi_n))(A)$. We shall prove the convexity of the complete moment set later on (see Proposition 3.2).

Now, if f is a \mathbb{C} -linear form on $\mathcal{U}(\mathfrak{g})$, and $g = \Re(f)$ its real part, then

$$f(A) = \Re(f)(A) - i\Re(f)(iA) = g(A) - ig(iA).$$

Conversely, if g is a \mathbb{R} -linear form on $\mathcal{U}(\mathfrak{g})$, it defines with the above formula a unique \mathbb{C} -linear mapping f such that $\Re(f) = g$.

Let us define the complex complete moment set $\tilde{J}(\pi)_{\mathbb{C}}$ of the representation (π, \mathcal{H}_π) of G in the same way as we did for $\tilde{J}(\pi)$, except that the elements f of $\tilde{J}(\pi)_{\mathbb{C}}$ are now complex valued linear functionals for which there is a sequence $(\xi_n) \in \mathcal{H}_\pi^\infty$, such that $\sum_{n \in \mathbb{N}} \|\xi_n\|_{\mathcal{H}_\pi}^2 = 1$ and $\sum_{n \in \mathbb{N}} \|d\pi(A)\xi_n\|_{\mathcal{H}_\pi}^2 < \infty$, ($A \in \mathcal{U}(\mathfrak{g})$). This sequence determines f by

$$f(A) := \tilde{\Psi}_{\pi, \mathbb{C}}((\xi_n))(A) := \sum_{n \in \mathbb{N}} \langle d\pi(A)\xi_n, \xi_n \rangle_{\mathcal{H}_\pi}, \quad \forall A \in \mathcal{U}(\mathfrak{g}). \quad (3)$$

Obviously, the equality of $\tilde{J}(\pi)$ and $\tilde{J}(\rho)$ is equivalent to $\tilde{J}(\pi)_{\mathbb{C}} = \tilde{J}(\rho)_{\mathbb{C}}$.

Let us also remark that:

Lemma 2.1. *Let π be a unitary representation of the connected Lie group G then*

$$J(\pi) \subset \tilde{J}(\pi) \subset \overline{\tilde{J}(\pi)}, \quad (4)$$

where $-$ denotes the closure in $\mathcal{U}(\mathfrak{g})^*$ for the topology of pointwise convergence.

Proof. $J(\pi) \subset \tilde{J}(\pi)$: Indeed if $\lambda_1, \dots, \lambda_k$ are positive numbers such that $\lambda_1 + \dots + \lambda_k = 1$ and ξ_1, \dots, ξ_n are C^∞ vectors with norm 1, we put $\xi'_n = \sqrt{\lambda_n} \xi_n$, for the (finite) sequence (ξ'_n) , $\sum_{n=1}^k \|\xi'_n\|_{\mathcal{H}_\pi}^2 = 1$ and

$$\tilde{\Psi}_\pi((\xi'_n))(A) = \Re \left(\frac{1}{i} \sum_{n=1}^k \langle d\pi(A)\xi'_n, \xi'_n \rangle_{\mathcal{H}_\pi} \right)$$

$$= \Re \left(\frac{1}{i} \sum_{n=1}^k \lambda_n \langle d\pi(A) \xi_n, \xi_n \rangle_{\mathcal{H}_\pi} \right) = \left(\sum_{n=1}^k \lambda_n \Psi_\pi(\xi_n) \right) (A).$$

$\tilde{J}(\pi) \subset \overline{J(\pi)}$: If f belongs to $\tilde{J}(\pi)$, then for all A in $\mathcal{U}(\mathfrak{g})$,

$$f(A) = \sum_{n=0}^{\infty} \Re \left(\frac{1}{i} \langle d\pi(A) \xi_n, \xi_n \rangle_{\mathcal{H}_\pi} \right); \quad \left(\sum_{n=0}^{\infty} \|\xi_n\|_{\mathcal{H}_\pi}^2 = 1 \right).$$

For any N in \mathbb{N} , we put

$$f_N(A) = \sum_{n=0}^N \Re \left(\frac{1}{i} \frac{\|\xi_n\|_{\mathcal{H}_\pi}^2}{\sum_{k=0}^N \|\xi_k\|_{\mathcal{H}_\pi}^2} \frac{\langle d\pi(A) \xi_n, \xi_n \rangle_{\mathcal{H}_\pi}}{\|\xi_n\|_{\mathcal{H}_\pi}^2} \right).$$

Then f_N is in $J(\pi)$ and

$$f_N(A) = \frac{1}{\sum_{k=0}^N \|\xi_k\|_{\mathcal{H}_\pi}^2} \sum_{n=0}^N \Re \left(\frac{1}{i} \langle d\pi(A) \xi_n, \xi_n \rangle_{\mathcal{H}_\pi} \right) \rightarrow f(A) \quad \text{as } N \text{ goes to } \infty.$$

Hence f is in $\overline{J(\pi)}$. \square

Let us now prove:

Lemma 2.2. *Let π and ρ be two unitary representations of G . If $J(\pi) = J(\rho)$, then $\tilde{J}(\pi) = \tilde{J}(\rho)$.*

Proof. Let f be in $\tilde{J}(\pi)$ and (ξ_n) a (finite or infinite) sequence in $\mathcal{H}_\pi^\infty \setminus \{0\}$ such that

$$\sum_n \|\xi_n\|_{\mathcal{H}_\pi}^2 = 1, \quad f(A) = \sum_n \Re \left(\frac{1}{i} \langle d\pi(A) \xi_n, \xi_n \rangle_{\mathcal{H}_\pi} \right), \quad A \in \mathcal{U}(\mathfrak{g}).$$

Let us put $\xi'_n = \frac{1}{\|\xi_n\|_{\mathcal{H}_\pi}} \xi_n$. The linear form f_n defined by

$$f_n(A) = \Re \left(\frac{1}{i} \langle d\pi(A) \xi'_n, \xi'_n \rangle_{\mathcal{H}_\pi} \right)$$

is for any n in $J(\pi) = J(\rho)$. There exists a finite sequence $\eta'_{K_{n-1}+1}, \dots, \eta'_{K_n}$ of norm 1 vectors in \mathcal{H}_ρ^∞ ($K_0 = 0$) and positive numbers $\mu_{K_{n-1}+1}, \dots, \mu_{K_n}$ such

that

$$\sum_{j=K_{n-1}+1}^{K_n} \mu_j = 1, \quad f_n(A) = \Re \left(\frac{1}{i} \sum_{j=K_{n-1}+1}^{K_n} \mu_j \langle d\rho(A) \eta'_j, \eta'_j \rangle_{\mathcal{H}_\rho} \right).$$

Put $\eta_j = \|\xi_n\|_{\mathcal{H}_\pi} \sqrt{\mu_j} \eta'_j$, ($j = 0, 1, \dots$) we get:

$$\|\xi_n\|_{\mathcal{H}_\pi}^2 f_n(A) = \Re \left(\frac{1}{i} \langle d\pi(A) \xi_n, \xi_n \rangle_{\mathcal{H}_\pi} \right) = \Re \left(\frac{1}{i} \sum_{j=K_{n-1}+1}^{K_n} \langle d\rho(A) \eta_j, \eta_j \rangle_{\mathcal{H}_\rho} \right)$$

and

$$\sum_{j=0}^{\infty} \|\eta_j\|_{\mathcal{H}_\rho}^2 = 1, \quad f(A) = \sum_{j=0}^{\infty} \Re \left(\frac{1}{i} \langle d\rho(A) \eta_j, \eta_j \rangle_{\mathcal{H}_\rho} \right).$$

This implies that for any $A \in \mathcal{U}(\mathfrak{g})$, if $A^+ \in \mathcal{U}(\mathfrak{g})$ is the formal adjoint of A ,

$$\sum_{j=0}^{\infty} \|d\rho(A) \eta_j\|_{\mathcal{H}_\rho}^2 = \sum_{j=0}^{\infty} \langle d\rho(A) \eta_j, d\rho(A) \eta_j \rangle_{\mathcal{H}_\rho} = f(iA^+A) < \infty.$$

This proves that $\tilde{J}(\pi) \subset \tilde{J}(\rho)$. The same argument gives the other inclusion. \square

Let us remark that we can have $J(\pi) = J(\rho)$ or $\tilde{J}(\pi) = \tilde{J}(\rho)$ for two non-equivalent unitary representations π and ρ . The simplest example is probably given by the trivial representation π and $\rho = \pi \oplus \pi$. For these two representations, all our moment sets coincide with $\{\varepsilon\}$ where ε is the augmentation map on $\mathcal{U}(\mathfrak{g})$: if (X_1, \dots, X_n) is a basis of \mathfrak{g} ,

$$\varepsilon(1) = 1, \quad \varepsilon(A) = 0, \quad \forall A \in \text{Span} \left(X_1^{i_1} \cdots X_n^{i_n}, \quad |i| = \sum_j i_j > 0 \right).$$

3. Moment sets and multiple of a representation

Let π be a unitary representation of the connected Lie group G on a separable Hilbert space \mathcal{H}_π . We define now the representation $\tilde{\pi} = \aleph_0 \pi$ (see [5] for instance).

We consider a sequence \mathcal{H}_n of Hilbert spaces unitary equivalent to \mathcal{H}_π through a unitary transformation $\psi_n : \mathcal{H}_\pi \rightarrow \mathcal{H}_n$. We define $\mathcal{H}_{\tilde{\pi}}$ as the Hilbert sum of the

spaces \mathcal{H}_n

$$\mathcal{H}_{\tilde{\pi}} = \oplus_{n \in \mathbb{N}} \mathcal{H}_n = \left\{ \tilde{\xi} = (\xi_n), \quad \xi_n \in \mathcal{H}_n, \quad \|\tilde{\xi}\|_{\mathcal{H}_{\tilde{\pi}}}^2 = \sum_n \|\xi_n\|_{\mathcal{H}_n}^2 < \infty \right\}$$

and the representation $\tilde{\pi}$ by

$$\tilde{\pi}(g) \left(\sum_{n \in \mathbb{N}} \xi_n \right) = \sum_{n \in \mathbb{N}} \psi_n \circ \pi(g) \circ \psi_n^{-1}(\xi_n), \quad g \in G.$$

It is easy to verify that $\tilde{\pi}$ is a continuous unitary representation of G .

Lemma 3.1. *Let π be a representation of G . Then $\tilde{\pi}$ and $\tilde{\tilde{\pi}}$ are unitary equivalent.*

The proof is an easy consequence of the well-known relation $\aleph_0^2 = \aleph_0$.

Now, following [5], we define the quasi-equivalence of representations.

Definition 3.1. Two unitary representations π and ρ of the Lie group G are quasi-equivalent (and we will write $\pi \sim \rho$) if the two representations $\tilde{\pi}$ and $\tilde{\rho}$ are equivalent.

Proposition 3.2. *Let π be a unitary representation of the connected Lie group G on a separable Hilbert space \mathcal{H}_π . Then*

$$\tilde{J}(\pi) = J(\tilde{\pi}).$$

Proof. First if a vector $\tilde{\xi} = \sum_n \psi_n(\xi_n)$ is a C^∞ vector for $\tilde{\pi}$, since the projection $p_n(\tilde{\xi}) = \xi_n$ is a continuous intertwining operator, ξ_n is a C^∞ vector for π . Moreover, if A is in $\mathcal{U}(\mathfrak{g})$, we get

$$d\tilde{\pi}(A)\tilde{\xi} = \sum_n \psi_n(d\pi(A)(\xi_n)) \quad \text{and} \quad \|d\tilde{\pi}(A)\tilde{\xi}\|_{\mathcal{H}_{\tilde{\pi}}}^2 = \sum_n \|d\pi(A)\xi_n\|_{\mathcal{H}_\pi}^2 < \infty.$$

Conversely, if (ξ_n) is a sequence of C^∞ vectors for π , such that, for any A in $\mathcal{U}(\mathfrak{g})$,

$$\sum_n \|d\pi(A)\xi_n\|_{\mathcal{H}_\pi}^2 < \infty,$$

then the vector $\tilde{\xi} = \sum_n \psi_n(\xi_n)$ is a C^∞ vector for $\tilde{\pi}$.

Indeed, if X is in \mathfrak{g} , the vector $\tilde{\eta} = \sum_n \psi_n(d\pi(X)\xi_n)$ is well defined in $\tilde{\mathcal{H}}_\pi$ and we can compute the derivative in 0 of

$$t \mapsto \tilde{\pi}(\exp tX)(\tilde{\xi}) = \sum_n \psi_n(\pi(\exp tX)\eta_n).$$

For any real numbers λ, t ,

$$\left| \frac{e^{it\lambda} - 1}{t} - i\lambda \right|^2 = \left(\frac{\cos(t\lambda) - 1}{t} \right)^2 + \left(\frac{\sin(t\lambda)}{t} - \lambda \right)^2 \leq 2\lambda^2.$$

Thus if $d\pi(X) = \int_{\mathbb{R}} i\lambda dE_\lambda$ is the spectral decomposition of $d\pi(X)$, we can write for any n :

$$\begin{aligned} \left\| \frac{\pi(\exp tX)(\xi_n) - \xi_n}{t} - d\pi(X)\xi_n \right\|_{\mathcal{H}_\pi}^2 &= \int_{\mathbb{R}} \left| \frac{e^{it\lambda} - 1}{t} - i\lambda \right|^2 d\langle E_\lambda(\xi_n), \xi_n \rangle_{\mathcal{H}_\pi} \\ &\leq \int_{\mathbb{R}} 2\lambda^2 d\langle E_\lambda(\xi_n), \xi_n \rangle_{\mathcal{H}_\pi} = 2 \|d\pi(X)\xi_n\|_{\mathcal{H}_\pi}^2. \end{aligned}$$

Then Lebesgue's dominated convergence theorem yields to

$$\lim_{t \rightarrow 0} \left\| \frac{\tilde{\pi}(\exp tX)(\tilde{\xi}) - \tilde{\xi}}{t} - \tilde{\eta} \right\|_{\tilde{\mathcal{H}}_\pi}^2 = \lim_{t \rightarrow 0} \sum_n \left\| \frac{\pi(\exp tX)(\xi_n) - \xi_n}{t} - d\pi(X)\xi_n \right\|_{\mathcal{H}_\pi}^2 = 0.$$

So $\tilde{\xi}$ is in the domain of $d\tilde{\pi}(X)$. By induction, we prove similarly, that it is in the domain of $d\tilde{\pi}(X_k \cdots X_1)$ for any X_i in \mathfrak{g} and

$$d\tilde{\pi}(X_k \cdots X_1)(\tilde{\xi}) = \sum_n \psi_n(d\pi(X_k \cdots X_1)(\xi_n)).$$

Hence

$$\begin{aligned} \mathcal{H}_\pi^\infty &= \left\{ \tilde{\xi} = \sum_n \psi_n(\xi_n), \quad \xi_n \in \mathcal{H}_\pi^\infty \right. \\ &\quad \left. \text{such that } \sum_n \|d\pi(A)\xi_n\|_{\mathcal{H}_\pi}^2 < \infty, \quad (A \in \mathcal{U}(\mathfrak{g})) \right\}. \end{aligned}$$

Now if f is in $\tilde{J}(\pi)_{\mathbb{C}}$, by definition, there is a sequence (ξ_n) of C^∞ vectors for π such that $f = \tilde{\Psi}_{\pi, \mathbb{C}}((\xi_n))$. We saw that $\tilde{\xi} = \sum_n \psi_n(\xi_n)$ is a C^∞ vector for $\tilde{\pi}$, with norm 1 and

$$\Psi_{\tilde{\pi}, \mathbb{C}}(\tilde{\xi})(A) = \langle d\tilde{\pi}(A)(\tilde{\xi}), \tilde{\xi} \rangle_{\mathcal{H}_{\tilde{\pi}}} = \sum_n \langle d\pi(A)(\xi_n), \xi_n \rangle_{\mathcal{H}_\pi} = \tilde{\Psi}_{\pi, \mathbb{C}}((\eta_n))(A).$$

Thus f belongs to $J(\tilde{\pi})_{\mathbb{C}}$.

Conversely, if f is in $J(\tilde{\pi})_{\mathbb{C}}$, there exists a finite sequence $\tilde{\xi}^0, \dots, \tilde{\xi}^{k-1}$ of C^∞ vectors for $\tilde{\pi}$ with norm 1, positive real numbers λ^j such that $\sum_{j=0}^{k-1} \lambda^j = 1$ and

$$f(A) = \sum_{j=0}^{k-1} \lambda^j \langle d\tilde{\pi}(A)\tilde{\xi}^j, \tilde{\xi}^j \rangle_{\mathcal{H}_{\tilde{\pi}}}.$$

Now, each vector $\tilde{\xi}^j$ can be written as $\sum_n \psi_n(\xi_n^j)$ with $\sum_n \|\xi_n^j\|_{\mathcal{H}_\pi}^2 = 1$ and $\sum_n \|d\pi(A)\xi_n^j\|_{\mathcal{H}_\pi}^2 < \infty$. Let us thus put: $\eta_{nk+r} = \sqrt{\lambda^r} \eta_n^r$ ($n = 0, 1, \dots$ and $0 \leq r < k$). Then

$$\sum_{p=0}^{\infty} \|\eta_p\|_{\mathcal{H}_\pi}^2 = \sum_{j=0}^{k-1} \lambda^j \sum_{n=0}^{\infty} \|\xi_n^j\|_{\mathcal{H}_\pi}^2 = 1$$

and

$$f(A) = \sum_{j=0}^{k-1} \lambda^j \langle d\tilde{\pi}(A)\tilde{\xi}^j, \tilde{\xi}^j \rangle_{\mathcal{H}_{\tilde{\pi}}} = \sum_{p=0}^{\infty} \frac{1}{i} \langle d\pi(A)\eta_p, \eta_p \rangle_{\mathcal{H}_\pi},$$

and f belongs to $\tilde{J}(\pi)_{\mathbb{C}}$. \square

4. Analytic vectors

Let us now recall briefly some well-known facts about analytic vectors for a unitary representation π of a Lie group G .

The reference for analytic vectors is [7]. Let (π, \mathcal{H}_π) be a unitary representation of the connected Lie group G . We say that a vector $\xi \in \mathcal{H}_\pi$ is analytic if the mapping

$$a_\xi : G \longrightarrow \mathcal{H}_\pi; \quad g \mapsto \pi(g)\xi$$

is analytic. Let \mathcal{H}_π^ω denote the (linear subspace) of analytic vectors of \mathcal{H}_π . By a result of Nelson (see [6]), the analytic vectors are dense in \mathcal{H}_π .

For any $\xi \in \mathcal{H}_\pi^\omega$, there exists a neighborhood U of 0 in the Lie algebra \mathfrak{g} of G , such that for any $X \in U$,

$$\pi(\exp X)\xi = \sum_{m=0}^{\infty} \frac{1}{m!} d\pi(X)^m \xi. \quad (5)$$

More precisely, if $\{X_1, \dots, X_p\}$ is a basis of \mathfrak{g} , then we have the following necessary and sufficient condition for a C^∞ -vector ξ in \mathcal{H}_π to be analytic. There exists a constant $M = M(\xi) > 0$, such that

$$\|(d\pi(X_{i_1}) \cdots d\pi(X_{i_m}))\xi\|_{\mathcal{H}_\pi} \leq M^m m! \quad \text{for all indices } i_1, \dots, i_m, \quad m \in \mathbb{N}. \quad (6)$$

This implies that if $\xi \in \mathcal{H}_\pi$ is analytic, then also the vector $\pi(g)\xi$ is analytic, since

$$\|(d\pi(X_{i_1}) \cdots d\pi(X_{i_m}))\pi(g)\xi\|_{\mathcal{H}_\pi} = \|(d\pi(\text{Ad}(g^{-1})X_{i_1}) \cdots d\pi(\text{Ad}(g^{-1})X_{i_m}))\xi\|_{\mathcal{H}_\pi}.$$

5. Proof of Theorem 1.2

First, if π and ρ are quasi-equivalent, $\tilde{\pi}$ and $\tilde{\rho}$ are unitarily equivalent, they thus have the same generalized moment set and by Proposition 3.2:

$$\tilde{J}(\pi) = J(\tilde{\pi}) = J(\tilde{\rho}) = \tilde{J}(\rho).$$

Conversely let us suppose that $\tilde{J}(\pi) = \tilde{J}(\rho)$, which implies that $\tilde{J}(\pi)_\mathbb{C} = \tilde{J}(\rho)_\mathbb{C}$. We first prove:

Lemma 5.1. *Suppose that π and ρ are two unitary representations of the connected Lie group G such that $\tilde{J}(\pi)_\mathbb{C} = \tilde{J}(\rho)_\mathbb{C}$. Let ξ be an analytic vector for the representation π with norm 1. Then there exists a sequence of analytic vectors $(\eta_n)_{n \in \mathbb{N}}$ for ρ such that for any g in G ,*

$$\langle \pi(g)\xi, \xi \rangle_{\mathcal{H}_\pi} = \sum_n \langle \rho(g)\eta_n, \eta_n \rangle_{\mathcal{H}_\rho}.$$

(Let us remark that if $g = 1$ in G , the relation means $\sum_n \|\eta_n\|_{\mathcal{H}_\rho}^2 = 1$ and thus the series converges absolutely for any g in G .)

Proof. Let us choose an analytic vector ξ in the space \mathcal{H}_π , with norm 1. Then the complex linear functional

$$f(A) := \langle d\pi(A)\xi, \xi \rangle_{\mathcal{H}_\pi}, \quad (A \in \mathcal{U}(\mathfrak{g}))$$

is an element of $\tilde{J}(\pi)_{\mathbb{C}}$. Hence, since $\tilde{J}(\pi)_{\mathbb{C}} = \tilde{J}(\rho)_{\mathbb{C}}$, there exists a sequence of C^∞ -vectors $(\eta_j)_j$ in \mathcal{H}_ρ^∞ , such that

$$\sum_{j \in \mathbb{N}} \|\eta_j\|_{\mathcal{H}_\rho}^2 = 1 \quad \text{and} \quad f(A) = \sum_{j \in \mathbb{N}} \langle d\rho(A)\eta_j, \eta_j \rangle_{\mathcal{H}_\rho}, \quad \forall A \in \mathcal{U}(\mathfrak{g}). \quad (7)$$

From this, if we take m in \mathbb{N} and $A = X_{i_1} \cdots X_{i_m}$ as in (6), it follows that

$$\begin{aligned} (M^m m!)^2 &\geq \|d\pi(A)\xi\|_{\mathcal{H}_\pi}^2 = \langle d\pi(A^+A)\xi, \xi \rangle_{\mathcal{H}_\pi} \\ &= f(A^+A) = \sum_{j \in \mathbb{N}} \langle d\rho(A^+A)\eta_j, \eta_j \rangle_{\mathcal{H}_\rho} = \sum_{j \in \mathbb{N}} \|d\rho(A)\eta_j\|_{\mathcal{H}_\rho}^2. \end{aligned}$$

Hence for every $j \in \mathbb{N}$, we have

$$\|d\rho(X_{i_1} \cdots X_{i_m})\eta_j\|_{\mathcal{H}_\rho} \leq M^m m! \quad (8)$$

and so η_j is an analytic vector for every $j \in \mathbb{N}$, moreover estimation (8) is independent of j .

Fix now $Y = \sum_{j=1}^p y_j X_j$ in \mathfrak{g} , define the norm of Y by

$$\|Y\| := \sup_{j \in \{1, \dots, p\}} |y_j|.$$

It follows from (6) that for $Y \in \mathfrak{g}$ and an analytic vector ξ in \mathcal{H}_π we have that

$$\|d\pi(Y)^m \xi\|_{\mathcal{H}_\pi} \leq p^m M(\xi)^m \|Y\|^m, \quad m \in \mathbb{N}. \quad (9)$$

Define the mapping $\varphi : \mathbb{R} \longrightarrow \mathcal{H}_\pi$ by

$$\varphi(t) := \pi(\exp tY)\xi.$$

Then

$$\varphi(t) = \sum_{k=0}^n \frac{t^k}{k!} d\pi(Y^k)\xi + R_n(t),$$

where

$$R_n(t) = \frac{1}{n!} \int_0^t (t-s)^n d\pi(Y)^{n+1} \varphi(s) ds$$

and so by (9), for $|t| < (p\|Y\|M(\xi))^{-1}$,

$$\begin{aligned}\|R_n(t)\|_{\mathcal{H}_\pi} &\leq \frac{1}{n!} \left| \int_0^t (t-s)^n \|d\pi(Y^{n+1})\varphi(s)\|_{\mathcal{H}_\pi} ds \right| \\ &\leq \|Y\|^{n+1} p^{n+1} M(\xi)^{n+1} |t|^{n+1} \rightarrow 0\end{aligned}$$

as n tends to ∞ .

Hence for all small t , the series $\sum_{k=0}^{\infty} \frac{1}{k!} \langle d\pi(tY)^k \xi, \xi \rangle_{\mathcal{H}_\pi}$ converges to $\langle \pi(\exp tY) \xi, \xi \rangle_{\mathcal{H}_\pi}$.

Furthermore, by (8), for every Y in the open ball $U := \left\{ Z \in \mathfrak{g}, \|Z\| < \frac{1}{pM(\xi)} \right\}$,

$$\rho(\exp Y)\eta_j = \sum_{k=0}^{\infty} \frac{1}{k!} d\rho(Y)^k \eta_j \quad (10)$$

converges absolutely and uniformly in j .

We also have absolute convergence for the double series $\sum_{k=0}^{\infty} \sum_j \frac{t^k}{k!} \langle d\rho(Y^k)\eta_j, \eta_j \rangle_{\mathcal{H}_\rho}$. Indeed, by (7), for Y in U and $|t| < 1$,

$$\begin{aligned}&\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{|t^k|}{k!} |\langle d\rho(Y^k)\eta_j, \eta_j \rangle_{\mathcal{H}_\rho}| \\ &\leq \sum_{k=0}^{\infty} \frac{|t^k|}{k!} \left(\sum_{n=0}^{\infty} \|d\rho(Y^k)\eta_j\|_{\mathcal{H}_\rho}^2 \right)^{\frac{1}{2}} \left(\sum_{j=0}^{\infty} \|\eta_j\|_{\mathcal{H}_\rho}^2 \right)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \frac{|t^k|}{k!} \|d\pi(Y^k)\xi\|_{\mathcal{H}_\pi} < \infty.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{j=0}^{\infty} \langle \rho(\exp tY)\eta_n, \eta_n \rangle_{\mathcal{H}_\rho} &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^k}{k!} \langle d\rho(Y^k)\eta_j, \eta_j \rangle_{\mathcal{H}_\rho} \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \langle d\pi(Y^k)\xi, \xi \rangle_{\mathcal{H}_\pi} = \langle \pi(\exp tY)\xi, \xi \rangle_{\mathcal{H}_\pi}.\end{aligned}$$

Let us define now the subset \mathcal{U} of G by,

$$\begin{aligned}\mathcal{U} &= \{g \in G, \text{ such that } \langle d\pi(A)\pi(g)\xi, \xi \rangle_{\mathcal{H}_\pi} \\ &= \sum_{j=0}^{\infty} \langle d\rho(A)\rho(g)\eta_j, \eta_j \rangle_{\mathcal{H}_\rho}, \quad \forall A \in \mathcal{U}(\mathfrak{g})\}.\end{aligned}$$

We shall now prove that $\mathcal{U} = G$. First, we show that \mathcal{U} is a closed subset of G . Let $(g_n)_n$ be a sequence of \mathcal{U} , which converges to g . Then, we have for all A in $\mathcal{U}(\mathfrak{g})$:

$$\begin{aligned}\langle d\pi(A)\pi(g)\xi, \xi \rangle_{\mathcal{H}_\pi} &= \langle \pi(g)\xi, d\pi(A^+)\xi \rangle_{\mathcal{H}_\pi} \\ &= \lim_{n \rightarrow \infty} \langle \pi(g_n)\xi, d\pi(A^+)\xi \rangle_{\mathcal{H}_\pi} \\ &= \lim_{n \rightarrow \infty} \sum_{j=0}^{\infty} \langle \rho(g_n)\eta_j, d\rho(A^+)\eta_j \rangle_{\mathcal{H}_\rho}.\end{aligned}$$

But

$$\begin{aligned}\sum_{j=0}^{\infty} |\langle \rho(g_n)\eta_j, d\rho(A^+)\eta_j \rangle_{\mathcal{H}_\rho}| &\leq \sum_{j=0}^{\infty} \|\rho(g_n)\eta_j\|_{\mathcal{H}_\rho} \|d\rho(A^+)\eta_j\|_{\mathcal{H}_\rho} \\ &\leq \left(\sum_j \|\rho(g_n)\eta_j\|_{\mathcal{H}_\rho}^2 \right)^{\frac{1}{2}} \left(\sum_j \|d\rho(A^+)\eta_j\|_{\mathcal{H}_\rho}^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_j \|\eta_j\|_{\mathcal{H}_\rho}^2 \right)^{\frac{1}{2}} \|d\pi(A^+)\xi\|_{\mathcal{H}_\pi} = \|d\pi(A^+)\xi\|_{\mathcal{H}_\pi} < \infty.\end{aligned}$$

Thus, by Lebesgue's theorem of dominated convergence,

$$\langle d\pi(A)\pi(g)\xi, \xi \rangle_{\mathcal{H}_\pi} = \sum_j \langle d\rho(A)\rho(g)\eta_j, \eta_j \rangle_{\mathcal{H}_\rho}.$$

Hence \mathcal{U} is closed.

Let us show that \mathcal{U} is open. Let g be in \mathcal{U} and A in $\mathcal{U}(\mathfrak{g})$. Hence, by 8, for any vector $Z = \sum_i z_i X_i$ in \mathfrak{g} such that $\|Z\| < 1$, we have

$$\|d\pi(Z)^k \xi\|_{\mathcal{H}_\pi} \leq (pM(\xi))^k k!, \quad k \in \mathbb{N}, \quad (11)$$

and so for all these Z 's and $|t| < (pM(\xi))^{-1}$, we obtain

$$\begin{aligned}I &:= \sum_{k=0}^{\infty} \sum_j \frac{|t^k|}{k!} |\langle d\rho(AZ^k)\rho(g)\eta_j, \eta_j \rangle_{\mathcal{H}_\rho}| \\ &= \sum_{k=0}^{\infty} \sum_j \frac{|t^k|}{k!} |\langle d\rho(Z^k)\rho(g)\eta_j, d\rho(A^+)\eta_j \rangle_{\mathcal{H}_\rho}|\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=0}^{\infty} \frac{|t^k|}{k!} \left(\sum_j \|d\rho(Z^k)\rho(g)\eta_j\|_{\mathcal{H}_\rho}^2 \right)^{\frac{1}{2}} \left(\sum_j \|d\rho(A^+)\eta_j\|_{\mathcal{H}_\rho}^2 \right)^{\frac{1}{2}} \\
&\leq \sum_{k=0}^{\infty} \frac{|t^k|}{k!} \|d\pi(Z^k)\pi(g)\xi\|_{\mathcal{H}_\pi} \|d\pi(A^+)\xi\|_{\mathcal{H}_\pi} < \infty.
\end{aligned}$$

This shows that for our Z , for $|t| < (pM(\xi))^{-1}$, since g is in \mathcal{U} , we have by (10)

$$\begin{aligned}
\langle d\pi(A)\pi(\exp tZ)\pi(g)\xi, \xi \rangle_{\mathcal{H}_\pi} &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \langle d\pi(AZ^k)\pi(g)\xi, \xi \rangle_{\mathcal{H}_\pi} \\
&= \sum_{k=0}^{\infty} \sum_j \frac{t^k}{k!} \langle d\rho(AZ^k)\rho(g)\eta_j, \eta_j \rangle_{\mathcal{H}_\rho} \\
&= \sum_j \langle d\rho(A)\rho(\exp tZ)\rho(g)\eta_j, \eta_j \rangle_{\mathcal{H}_\rho}
\end{aligned}$$

or

$$\langle d\pi(A)\pi(\exp tZg)\xi, \xi \rangle_{\mathcal{H}_\pi} = \sum_j \langle d\rho(A)\rho(\exp tZg)\eta_j, \eta_j \rangle_{\mathcal{H}_\rho}.$$

This shows that \mathcal{U} is an open set. Hence, since G is connected, $G = \mathcal{U}$. Especially, for every g in G , we have

$$\langle \pi(g)\xi, \xi \rangle_{\mathcal{H}_\pi} = \sum_j \langle \rho(g)\eta_j, \eta_j \rangle_{\mathcal{H}_\rho},$$

the series converging absolutely. \square

Let us present now the proof of Theorem 1.2

Let us consider the set \mathcal{V} of all the families $V = (\xi_i)_{i \in I}$ of analytic vectors for π , with norm 1 and such that, if $i \neq j$, then $\langle \pi(g)\xi_i, \xi_j \rangle_{\mathcal{H}_\pi} = 0$ for any g in G . On \mathcal{V} , we consider the ordering given by inclusion. We get an inductive set (\mathcal{V}, \subset) . By Zorn's lemma, there is a family W in \mathcal{V} which is maximal.

Since the vectors ξ_i in W form an orthogonal system in the separable space \mathcal{H}_π , W is finite or countable. We suppose now the vectors in W labelled by integers $W = (\xi_0, \xi_1, \dots)$.

For each n in \mathbb{N} , we note \mathcal{K}_n the closure of the vector space spanned by all the $\pi(g)\xi_n$ (g in G). The spaces \mathcal{K}_n are mutually orthogonal subspaces of \mathcal{H}_π , since,

if $n \neq p$,

$$\left\langle \sum_{i=1}^k \lambda_i \pi(g_i) \xi_n, \sum_{\ell=1}^q \mu_\ell \pi(g'_\ell) \xi_p \right\rangle_{\mathcal{H}_\pi} = \sum_{i,\ell} \bar{\lambda}_i \mu_\ell \left\langle \pi((g'_\ell)^{-1} g_i) \xi_n, \xi_p \right\rangle_{\mathcal{H}_\pi} = 0.$$

We now consider the Hilbert sum \mathcal{K} of the \mathcal{K}_n as a closed subspace of \mathcal{H}_π . In fact this subspace coincides with \mathcal{H}_π . Indeed, each \mathcal{K}_n being a G -invariant subspace of \mathcal{H}_π , \mathcal{K} itself is invariant. If $\mathcal{K} \neq \mathcal{H}_\pi$, the non-trivial invariant subspace \mathcal{K}^\perp contains an analytic vector η with norm 1, $W \cup \{\eta\}$ is in \mathcal{V} and larger than W . This is impossible and so $\oplus_n \mathcal{K}_n = \mathcal{H}_\pi$.

For each ξ_n , according to Lemma 5.1, there exist analytic vectors $\eta_{n,k}$ for ρ such that

$$\langle \pi(g) \xi_n, \xi_n \rangle_{\mathcal{H}_\pi} = \sum_{k=0}^{\infty} \langle \rho(g) \eta_{n,k}, \eta_{n,k} \rangle_{\mathcal{H}_\rho} \quad (g \in G).$$

Now we can build an intertwining operator T_n between $\pi|_{\mathcal{K}_n}$ and $\tilde{\rho}$.

First, we note $\mathcal{H}_{\tilde{\rho}} = \oplus_k \theta_k(\mathcal{H}_\rho)$ and define T_n on linear combinations $\sum_{i=1}^{\ell} \lambda_i \pi(g_i) \xi_n$ by

$$T_n \left(\sum_{i=1}^{\ell} \lambda_i \pi(g_i) \xi_n \right) = \sum_{k=0}^{\infty} \sum_{i=1}^{\ell} \lambda_i \theta_k(\rho(g_i) \eta_{n,k}).$$

For the moment, T_n is not well defined with this formula, but we have

$$\begin{aligned} \left\| \sum_{k=0}^{\infty} \sum_{i=1}^{\ell} \theta_k(\rho(g_i) \eta_{n,k}) \right\|_{\mathcal{H}_\rho}^2 &= \sum_k \left\langle \sum_i \lambda_i \rho(g_i) \eta_{n,k}, \sum_j \lambda_j \rho(g_j) \eta_{n,k} \right\rangle_{\mathcal{H}_\rho} \\ &= \sum_k \sum_{i,j} \bar{\lambda}_i \lambda_j \langle \rho(g_j^{-1} g_i) \eta_{n,k}, \eta_{n,k} \rangle_{\mathcal{H}_\rho} \\ &= \sum_{i,j} \bar{\lambda}_i \lambda_j \sum_k \langle \rho(g_j^{-1} g_i) \eta_{n,k}, \eta_{n,k} \rangle_{\mathcal{H}_\rho} \\ &= \sum_{i,j} \bar{\lambda}_i \lambda_j \langle \pi(g_j^{-1} g_i) \xi_n, \xi_n \rangle_{\mathcal{H}_\pi} \\ &= \left\| \sum_i \lambda_i \pi(g_i) \xi_n \right\|_{\mathcal{H}_\pi}^2. \end{aligned}$$

Note that the permutation of the sums on k and i, j are allowed since all the series in k are absolutely convergent.

This relation proves that T_n is well defined since if $\sum_i \lambda_i \pi(g_i) \xi_n = 0$, then

$$T_n \left(\sum_i \lambda_i \pi(g_i) \xi_n \right) = 0,$$

moreover T_n is isometric. Thus T_n can be uniquely extended to \mathcal{K}_n by continuity.

By definition, T_n is an intertwining operator between $\pi|_{\mathcal{K}_n}$ and $\tilde{\rho}$

$$\begin{aligned} (T_n \circ \pi(g)) \left(\sum_i \lambda_i \pi(g_i) \xi_n \right) &= T_n \left(\sum_i \lambda_i \pi(g g_i) \xi_n \right) \\ &= \sum_k \sum_i \lambda_i \theta_k(\rho(g g_i) \eta_{n,k}) \\ &= (\tilde{\rho}(g) \circ T_n) \left(\sum_i \lambda_i \pi(g_i) \xi_n \right). \end{aligned}$$

Consider now the mapping $T = \oplus_n T_n$. By construction, it is a partial isometry between \mathcal{H}_π and $\mathcal{H}_{\tilde{\rho}}$. With this construction, we get an intertwining operator between π and $\tilde{\tilde{\rho}}$. But Lemma 3.1 gives us a unitary intertwining U between $\tilde{\tilde{\rho}}$ and $\tilde{\rho}$ and $S = U \circ T$ is a partial isometry intertwining π and $\tilde{\rho}$.

Now we can define $\tilde{S} : \mathcal{H}_{\tilde{\pi}} = \oplus \psi_n(\mathcal{H}_\pi) \longrightarrow \mathcal{H}_{\tilde{\rho}} = \oplus \tilde{\theta}_n(\mathcal{H}_{\tilde{\rho}})$ by

$$\tilde{S} \left(\sum_{n=0}^{\infty} \psi_n(v_n) \right) = \sum_{n=0}^{\infty} \tilde{\theta}_n(S(v_n)), \quad v_n \in \mathcal{H}_\pi, n \in \mathbb{N}.$$

Finally the partial isometry $\tilde{V} = U \circ \tilde{S}$ intertwines $\tilde{\pi}$ and $\tilde{\rho}$.

Exchanging the roles of π and ρ yields to a partial isometry intertwining $\tilde{\rho}$ and $\tilde{\pi}$.

With the notations of [5], we thus have $\tilde{\pi} \leq \tilde{\rho}$ and $\tilde{\rho} \leq \tilde{\pi}$. Hence, with Corollary 5.1.5 of [5], we conclude that $\tilde{\pi}$ and $\tilde{\rho}$ are unitarily equivalent. \square

6. Proof of Theorem 1.3

We are now going to prove Theorem 1.3. Let π and ρ be two irreducible unitary representations of the connected Lie group G . It is clear that if π and ρ are unitarily equivalent ($\pi \simeq \rho$), then every coefficient of π is also a coefficient of ρ and so $J(\pi)$ and $J(\rho)$ coincide, we saw that $J(\pi) = J(\rho)$ implies $\tilde{J}(\pi) = \tilde{J}(\rho)$ and $\tilde{J}(\pi)_{\mathbb{C}} = \tilde{J}(\rho)_{\mathbb{C}}$.

Hence it suffices to show that $\tilde{J}(\pi)_{\mathbb{C}} = \tilde{J}(\rho)_{\mathbb{C}}$ implies that π and ρ are equivalent.

We use Lemma 5.1. Let ξ be an analytic vector for π with norm 1. There is a sequence (η_n) of analytic vectors for ρ such that

$$\langle \pi(g)\xi, \xi \rangle_{\mathcal{H}_\pi} = \sum_n \langle \rho(g)\eta_n, \eta_n \rangle_{\mathcal{H}_\rho} \quad (g \in G).$$

From now on, we suppose, without loss of generality, that $\eta_0 \neq 0$.

We saw that $\tilde{\rho}$ is a unitary representation of G and the linear mapping P from $\mathcal{H}_{\tilde{\rho}}$ to \mathcal{H}_ρ , defined by

$$P \left(\sum_n \theta_n(\zeta_n) \right) = \zeta_0$$

is a non-trivial intertwining operator between $\tilde{\rho}$ and ρ :

$$(P \circ \tilde{\rho}(g)) \left(\sum_n \theta_n(\zeta_n) \right) = \rho(g)\zeta_0 = (\rho(g) \circ P)(g) \left(\sum_n \theta_n(\zeta_n) \right).$$

Now as in the proof of Theorem 1.2, there is an isometric intertwining operator T between π and $\tilde{\rho}$ defined by

$$T \left(\sum_{i=1}^k \lambda_i \pi(g_i) \xi \right) = \sum_{n=0}^{\infty} \sum_{i=1}^k \theta_n(\rho(g_i)\eta_n).$$

The representation π being irreducible and the linear space V generated by the vectors $\pi(g)\xi$ invariant, this space is dense in \mathcal{H}_π . Thus T can be uniquely extended to \mathcal{H}_π by continuity. By definition, T is an intertwining between π and $\tilde{\rho}$, since for vectors in V ,

$$\begin{aligned} (T \circ \pi(g)) \left(\sum_i \lambda_i \pi(g_i) \xi \right) &= T \left(\sum_i \lambda_i \pi(gg_i) \Phi \right) \\ &= \sum_n \sum_i \lambda_i \theta_n(\rho(gg_i)\eta_n) \\ &= (\tilde{\rho}(g) \circ T) \left(\sum_i \lambda_i \pi(g_i) \xi \right). \end{aligned}$$

Now $A = P \circ T$ is an intertwining operator between π and ρ , since $A\xi = \eta_0 \neq 0$, it is non-trivial. Hence by Warner [7] Corollary 4.3.1.3, the irreducible representations π and ρ are unitarily equivalent. This proves our result.

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